Published by Institute of Physics Publishing for SISSA
Received: *February 25, 2007*
Accepted: *April 5, 2007*
Published: *April 18, 2007*

Linearizing generalized Kähler geometry

Ulf Lindström,[ab] Martin Roček,[c] Rikard von Unge[d] and Maxim Zabzine[a]

[a]*Department of Theoretical Physics Uppsala University*
Box 803, SE-751 08 Uppsala, Sweden
[b]*HIP-Helsinki Institute of Physics, University of Helsinki*
P.O. Box 64 FIN-00014 Suomi-Finland
[c]*C.N. Yang Institute for Theoretical Physics, Stony Brook University*
Stony Brook, NY 11794-3840,U.S.A.
[d]*Institute for Theoretical Physics, Masaryk University*
61137 Brno, Czech Republic
E-mail: Ulf.Lindstrom@teorfys.uu.se, rocek@insti.physics.sunysb.edu, unge@physics.muni.cz, Maxim.Zabzine@teorfys.uu.se

Abstract: The geometry of the target space of an $N = (2, 2)$ supersymmetry sigma-model carries a generalized Kähler structure. There always exists a real function, the generalized Kähler potential K, that encodes all the relevant local differential geometry data: the metric, the B-field, *etc.* Generically this data is given by *nonlinear* functions of the second derivatives of K. We show that, at least locally, the nonlinearity on any generalized Kähler manifold can be explained as arising from a quotient of a space without this nonlinearity.

Keywords: Superspaces, Sigma Models, Differential and Algebraic Geometry, Extended Supersymmetry.

Contents

JHEP04(2007)061

1. Introduction

This paper is an elaboration and continuation of our previous paper [12], where we proved the existence of the generalized Kähler potential K and solved the problem of $N = (2, 2)$ off-shell supersymmetry for sigma-models. For a generic generalized Kähler manifold, all geometrical data such as the metric g, the B-field and the complex structures are expressible in terms of derivatives of K. In the generic case K enters nonlinearly. In the present article we show that this nonlinearity arises from a quotient construction. Indeed, locally, any generalized Kähler manifold may be thought of as a quotient of another generalized Kähler manifold with very special properties.

At the level of the $N = (2, 2)$ sigma-models our idea is simple: If the model contains only (anti)chiral and twisted (anti)chiral superfields, the left and right complex structures commute, and *all geometrical data on the target space is expressed linearly in terms of the generalized Kähler potential*. We call such target spaces bihermitian local product spaces (BiLPs). All nonlinearity is related to the semichiral fields. We show that by gauging certain symmetries, an appropriate combination of chiral and twisted chiral fields may be traded for semichiral fields. Thus any $N = (2, 2)$ sigma-model can be understood as a quotient of an $N = (2, 2)$ sigma-model containing only chiral and twisted chiral fields, i.e., defined over a specific BiLP space (the auxiliary local product or ALP space). In this paper we describe the properties of the ALP space and present the details of the quotient. We attempt to present the natural sigma-model construction in geometrical terms.

Mathematically we can formulate our results as follows: Locally, for any generalized Kähler manifold M we consider an auxiliary space $M \times \mathbb{C}^{2d_s}$, where $2d_s$ is the dimension of the cokernel of the commutator of the left and right complex structures on M. This auxiliary space is also a generalized Kähler manifold but with the additional property that the two complex structures commute. The fiber $\mathbb{C}^{2d_s}$ is spanned by those vectors that respect the underlying generalized Kähler geometry. All structures on M can be understood via the quotient construction described in this paper.

The paper is organized as follows. In section 2 we review the basic definitions and properties of generalized Kähler geometry. In particular we discuss the special class of these geometries where the generalized Kähler potential enters linearly. Section 3 recasts the geometry in terms of $N = (2, 2)$ sigma-models and sets up the notation for further discussion. Section 4 presents the central idea of this paper at the level of $N = (2, 2)$ sigma-models, namely, we show that any model with semichiral fields can be thought of as a quotient of a model that contains only chiral and twisted chiral fields. The remaining sections are devoted to the geometrical explanation of this statement. Section 5 describes ALPs (auxiliary local product spaces), which are built over any generalized Kahler manifold as a principal bundle with a free action of abelian isometries. Section 6 presents various quotient constructions used in the context of sigma-models; in particular, we explain how to perform the quotient on the ALPs. Section 7 contains some speculations about the possibility of a global version of the quotient. Section 8 includes a summary of the results and some open problems. We also include three appendices in which we discuss possible potentials on generalized Kähler manifolds, properties of the ALP space generalized Kähler potential, and the explicit calculations of the quotient data.

2. Generalized Kähler geometry

In this section, we present some geometrical background; we mostly review well-known facts, but present some new observations as well.

2.1 General definition and properties

The definition of (twisted) generalized Kähler geometry originates in the study of the general $N = (2, 2)$ supersymmetric sigma-models [3], although the name and the renewed interest is due to recent work of Gualtieri [5].

We define (twisted) generalized Kähler geometry[1] (M, J_+, J_-, g, H) as the following data on a smooth manifold M: $J_\pm$ are two complex structures and g is a metric which is bihermitian

$$J_\pm^t g J_\pm = g \ . \tag{2.1}$$

Moreover the complex structures $J_\pm$ are covariantly constant

$$\nabla^{(\pm)} J_\pm = 0 \tag{2.2}$$

with respect to the connections with torsion

$$\Gamma^\pm = \Gamma \pm g^{-1} H \, , \tag{2.3}$$

where Γ is the Levi-Civita connection and H is a closed three-from. The name "generalized Kähler geometry" is motivated by the fact that when $J_+ = \pm J_-$ we recover standard Kähler geometry.

Alternatively, we can define a generalized Kähler geometry (M, J_+, J_-, g) as two complex structures $J_\pm$ with a bihermitian metric g (2.1) and the integrability conditions

$$d_+^c \omega_+ + d_-^c \omega_- = 0 \, , \qquad d d_\pm^c \omega_\pm = 0 \, , \tag{2.4}$$

where $\omega_\pm \equiv g J_\pm$ and $d_\pm^c$ are the $i(\bar{\partial} - \partial)$ operators associated to the complex structures $J_\pm$. The corresponding closed three-form that gives the torsion in the connections (2.3) is defined as

$$H = d_+^c \omega_+ = -d_-^c \omega_- \, , \tag{2.5}$$

and is not an independent geometrical datum.

The generalized Kähler manifold (M, J_+, J_-, g) admits three different Poisson structures: two real Poisson structures $\pi_\pm = (J_+ \pm J_-) g^{-1}$ [13] and the holomorphic Poisson structure $\sigma = [J_+, J_-] g^{-1}$ [7] with the following obvious relation between the kernels

$$\ker \sigma = \ker \pi_+ \oplus \ker \pi_- \ . \tag{2.6}$$

We call $x_0 \in M$ a regular point of the generalized Kähler manifold if the ranks of $\pi_\pm$ do not vary in an open neighborhood of x_0. All other points of M are called singular points. The set of regular points of M is obviously open and is dense in M. If all points of M are regular we call such an M a regular generalized Kähler manifold.

Theorem 1. *A generalized Kähler manifold with $H = 0$ is regular.*

Proof: This follows immediately from Theorem 2.20 in [18] and the fact that when $H = 0$, the Poisson structures $\pi_\pm$ are covariantly constant with respect to the Levi-Civita connection. □

[1] Motivated by generalized complex geometry, typically the word "twisted" is added when $H \neq 0$. Here we omit "twisted" and assume that $H \neq 0$ unless otherwise stated.

In the general case, in a neighborhood of a regular point we can introduce coordinates along the symplectic foliations of $\pi_\pm$ and σ; see [12] for more details. These local coordinates are adapted to the following decomposition

$$\ker(J_+ - J_-) \oplus \ker(J_+ + J_-) \oplus \mathrm{coker}[J_+, J_-]\,, \tag{2.7}$$

where (the real) $\dim(\mathrm{coker}[J_+, J_-])$ is a multiple of four, as it corresponds to a symplectic leaf of the holomorphic Poisson structure σ, and $\dim(\ker(J_+ \pm J_-))$ is a multiple of two.

The generalized Kähler geometry can be nicely described in the context of the generalized geometric structures introduced by Hitchin [6]: a generalized Kähler geometry is a pair of commuting (twisted) generalized complex structures $\mathcal{J}_{1,2}$ such that their product induces a definite metric on $TM \oplus T^*M$. The equivalence of this definition with the one presented before was proven in [5]; see [1] for an alternative explanation of this result via sigma-models. The following relation holds between the type of (twisted) generalized complex structures (see [5] for the definition) and kernels of $\pi_\pm$,

$$\mathsf{type}(\mathcal{J}_{1,2}) = \frac{1}{2}\dim(\ker \pi_\pm) = \frac{1}{2}\dim(\ker(J_+ \pm J_-))\ . \tag{2.8}$$

Therefore a regular generalized Kähler manifold can be defined as one where the type of $\mathcal{J}_{1,2}$ is constant, i.e., no change in type occurs, and hence all untwisted generalized Kähler manifolds with $H = 0$ are regular (see the theorem above).

2.2 The generalized Kähler potential

In this subsection we review the arguments from [12] concerning the existence of a generalized Kähler potential. We also comment on symmetries of K.

Consider a neighborhood of a regular point x_0 of a generalized Kähler manifold and choose local coordinates adapted to the symplectic foliation of σ. We can choose coordinates $\{q, p, z, z'\}$ in which J_+ has the canonical form

$$J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix}\,, \tag{2.9}$$

where a collective notation is used in the matrices, and where J_c, J_t, and J_s are $2d_c$, $2d_t$, and $2d_s$ dimensional canonical complex structures of the form

$$J = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\ . \tag{2.10}$$

The coordinates z and z' parametrize the kernels of $\pi_\mp$ respectively, and $\{q, p\}$ are the Darboux coordinates for a symplectic leaf of σ. The subscript "c" (chiral) corresponds to the coordinates along the kernel of π_-, the subscript "t" (twisted chiral) corresponds to the coordinates along the kernel of π_+ and "s" (semichiral) denotes the coordinates along the leaf of σ.

JHEP04(2007)061

Alternatively we can choose the coordinates $\{Q, P, z, z'\}$ in which J_- has a canonical form

$$J_- = \begin{pmatrix} J_s & 0 & 0 & 0 \\ 0 & J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix} . \qquad (2.11)$$

Again (Q, P) are the Darboux coordinates on a leaf of σ.

The coordinates $\{q, p\}$ are related to $\{Q, P\}$ by a canonical transformation and we may thus introduce the corresponding generating function. Choosing new coordinates $\{q, P\}$ along a leaf in a neighborhood of x_0, there exists a family of generating functions $K(q, P, z, z')$ such that

$$p = \frac{\partial K}{\partial q} , \qquad Q = \frac{\partial K}{\partial P} \qquad (2.12)$$

is satisfied. So far $K(q, P, z, z')$ is defined up to the addition of an arbitrary function $F(z, z')$, which will be partially fixed later on. We can also shift K by a J_+-holomorphic function $f(q, z, z')$ plus its complex conjugate function, as this just gives a J_+-holomorphic redefinition of the complex coordinates $\{q, p, z, z'\}$. Analogously we can shift K by a J_--holomorphic function $g(P, z, z')$ plus its complex conjugate; this preserves J_-. Thus these additional shifts are also natural symmetries of our problem.

Moreover, since K is a generating function, it is natural to consider its Legendre transforms. For example, if we would like to switch from $\{q, P, z, z'\}$ to $\{p, Q, z, z'\}$, then the corresponding generating function $\tilde{K}(p, Q, z, z')$ is a Legendre transform[2] of $K(q, P, z, z')$

$$\tilde{K} = K - pq - QP \qquad (2.13)$$

with (2.12) taken into account. Similar constructions relate the other possible generating functions.

Next, using (2.12), we can perform a coordinate transformations from $\{q, p, z, z'\}$ to $\{q, P, z, z'\}$ and calculate J_+

$$J_+ = \begin{pmatrix} J_s & 0 & 0 & 0 \\ K_{RL}^{-1} C_{LL} & K_{RL}^{-1} J_s K_{LR} & K_{RL}^{-1} C_{Lc} & K_{RL}^{-1} C_{Lt} \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & J_t \end{pmatrix} . \qquad (2.14)$$

The subscript "L" stands for q, subscript "R" stands for P, the subscript c for z and the subscript t is for z'. K_{LR}, C_{LL} and A_{LL}, *etc.*, are included in the set of submatrices defined by

$$K_{LR} \equiv \begin{pmatrix} K_{ab'} & K_{a\bar{b}'} \\ K_{\bar{a}b'} & K_{\bar{a}\bar{b}'} \end{pmatrix} \qquad K_{LL} \equiv \begin{pmatrix} K_{ab} & K_{a\bar{b}} \\ K_{\bar{a}b} & K_{\bar{a}\bar{b}} \end{pmatrix} , \qquad (2.15)$$

[2]This Legendre transform exists if $\{p, Q, z, z'\}$ are good coordinates around x_0.

and similarily for K_{RR}, as well as

$$C \equiv JK - KJ = \begin{pmatrix} 0 & 2iK \\ -2iK & 0 \end{pmatrix},$$

$$A \equiv JK + KJ = \begin{pmatrix} 2iK & 0 \\ 0 & -2iK \end{pmatrix}, \tag{2.16}$$

where the different possible subscripts have been suppressed. In (2.15) we use the notation $K_{ab} \equiv \partial_a \partial_b K$, *etc.*, and define $K_{LR}^{-1} \equiv (K_{RL})^{-1}$.

Analogously, using (2.12), we can perform the coordinate transformations from $\{Q, P, z, z'\}$ to $\{q, P, z, z'\}$ and calculate J_-. The result is

$$J_- = \begin{pmatrix} K_{LR}^{-1} J_s K_{RL} & K_{LR}^{-1} C_{RR} & -K_{LR}^{-1} C_{Rc} & K_{LR}^{-1} A_{Rt} \\ 0 & -J_s & 0 & 0 \\ 0 & 0 & J_c & 0 \\ 0 & 0 & 0 & -J_t \end{pmatrix}. \tag{2.17}$$

We now turn to the remaining data in our model: the metric g and the closed three-form H. The latter may be locally expressed in terms of a two-form B such that $H = dB$. It is then convenient to define $E \equiv \frac{1}{2}(g + B)$.

If $\sigma \equiv [J_+, J_-] g^{-1}$ is invertible, $E = J_+ J_- \sigma^{-1}$; more generally, equations (2.4) and (2.5) provide differential equations for g and B which may be solved knowing $J_\pm$ and remembering that $\omega_\pm \equiv g J_\pm$. The solution may again be expressed in terms of the submatrices introduced in (2.15) and (2.16). The solution is [12]

$$\begin{aligned}
E_{LL} &= C_{LL} K_{LR}^{-1} J_s K_{RL} \\
E_{LR} &= J_s K_{LR} J_s + C_{LL} K_{LR}^{-1} C_{RR} \\
E_{Lc} &= K_{Lc} + J_s K_{Lc} J_c + C_{LL} K_{LR}^{-1} C_{Rc} \\
E_{Lt} &= -K_{Lt} - J_s K_{Lt} J_t + C_{LL} K_{LR}^{-1} A_{Rt} \\
E_{RL} &= -K_{RL} J_s K_{LR}^{-1} J_s K_{RL} \\
E_{RR} &= -K_{RL} J_s K_{LR}^{-1} C_{RR} \\
E_{Rc} &= K_{Rc} - K_{RL} J_s K_{LR}^{-1} C_{Rc} \\
E_{Rt} &= -K_{Rt} - K_{RL} J_s K_{LR}^{-1} A_{Rt} \\
E_{cL} &= C_{cL} K_{LR}^{-1} J_s K_{RL} \\
E_{cR} &= J_c K_{cR} J_s + C_{cL} K_{LR}^{-1} C_{RR} \\
E_{cc} &= K_{cc} + J_c K_{cc} J_c + C_{cL} K_{LR}^{-1} C_{Rc} \\
E_{ct} &= -K_{ct} - J_c K_{ct} J_t + C_{cL} K_{LR}^{-1} A_{Rt} \\
E_{tL} &= C_{tL} K_{LR}^{-1} J_s K_{RL} \\
E_{tR} &= J_t K_{tR} J_s + C_{tL} K_{LR}^{-1} C_{RR} \\
E_{tc} &= K_{tc} + J_t K_{tc} J_c + C_{tL} K_{LR}^{-1} C_{Rc} \\
E_{tt} &= -K_{tt} - J_t K_{tt} J_t + C_{tL} K_{LR}^{-1} A_{Rt}
\end{aligned} \tag{2.18}$$

Thus we have locally expressed the generalized Kähler geometry in terms of a single function K, the generalized Kähler potential. This potential is not uniquely defined, as follows from

JHEP04(2007)061

our previous discussion: It can be Legendre transformed and shifted by (the real part of) a $J_\pm$-holomorphic function.

2.3 Bihermitian local product geometry

In this subsection we draw attention to a special subset of generalized Kähler manifolds. We refer to a generalized Kähler geometry with the additional property $[J_+, J_-] = 0$ as a bihermitian local product (BiLP) geometry. For BiLP geometries, g and H are linear in the generalized Kähler potential.

Equivalently, in the context of generalized geometry, we can define BiLP geometry as a generalized Kähler geometry with the addditional condition

$$\mathsf{type}(\mathcal{J}_1) + \mathsf{type}(\mathcal{J}_2) = \frac{1}{2}\dim M\,, \tag{2.19}$$

where $\dim M$ is the real dimension of M. The BiLP manifolds are examples of regular generalized Kähler manifolds.[3]

Many important properties of BiLP geometry were already pointed out in [3]. In particular, there exists a local product structure $\Pi = J_+J_-$, which induces a decomposition of TM into ± 1-eigenspaces. This decomposition into ± 1-pieces carries over to the differential forms

$$\Omega(M) = \bigoplus_{l+m=d} \Omega^{l,m}(M)\ . \tag{2.20}$$

Furthermore there is a compatible decomposition with respect to J_+, i.e., each ± 1 piece decomposes into holomorphic and antiholomorphic pieces correspondingly. Thus we have the following decomposition of the differential forms

$$\Omega(M) = \bigoplus_{p+q+n+r=d} \Omega^{p,q,n,r}(M)\ . \tag{2.21}$$

This decomposition gives rise to a decomposition of the exterior derivative

$$d = \partial_\phi + \bar\partial_\phi + \partial_\chi + \bar\partial_\chi \tag{2.22}$$

in terms of four mutually anti-commuting differentials. Thus (ϕ, χ) are J_+-holomorphic coordinates, $z \equiv (\phi, \bar\phi)$ parametrize $\ker(J_+ - J_-)$, and $z' \equiv (\chi, \bar\chi)$ parametrize $\ker(J_+ + J_-)$.

Now we can solve the conditions (2.4) locally. The condition (2.5) becomes

$$H = i(\bar\partial_\phi + \bar\partial_\chi - \partial_\phi - \partial_\chi)\omega_+ = -i(\bar\partial_\phi + \partial_\chi - \partial_\phi - \bar\partial_\chi)\omega_-\,, \tag{2.23}$$

which implies

$$\partial_\phi(\omega_+ + \omega_-) = 0\,, \quad \bar\partial_\phi(\omega_+ + \omega_-) = 0\,, \quad \partial_\chi(\omega_+ - \omega_-) = 0\,, \quad \bar\partial_\chi(\omega_+ - \omega_-) = 0\ . \tag{2.24}$$

These equations can be solved locally in terms of two real functions $K_1(\phi, \bar\phi, \chi, \bar\chi)$ and $K_2(\phi, \bar\phi, \chi, \bar\chi)$

$$\omega_+ = i\partial_\phi\bar\partial_\phi K_1 + i\partial_\chi\bar\partial_\chi K_2\,, \qquad \omega_- = i\partial_\phi\bar\partial_\phi K_1 - i\partial_\chi\bar\partial_\chi K_2\ . \tag{2.25}$$

[3] A global example of a BiLP structure can be found on the group manifold $SU(2) \times U(1)$; see [14, 15].

However, from (2.23) and $dH = 0$ these two functions are related by

$$\partial_\phi \bar{\partial}_\phi \partial_\chi \bar{\partial}_\chi K_1 + \partial_\phi \bar{\partial}_\phi \partial_\chi \bar{\partial}_\chi K_2 = 0 \tag{2.26}$$

and therefore their sum can be written in the form

$$K_1 + K_2 = f(\phi, \bar{\phi}, \chi) + \bar{f}(\bar{\phi}, \phi, \bar{\chi}) + g(\phi, \chi, \bar{\chi}) + \bar{g}(\bar{\phi}, \bar{\chi}, \chi) \ . \tag{2.27}$$

Thus we may define a single function K such that

$$K_1 = K + g(\phi, \chi, \bar{\chi}) + \bar{g}(\bar{\phi}, \bar{\chi}, \chi), \qquad K_2 = -K + f(\phi, \bar{\phi}, \chi) + \bar{f}(\bar{\phi}, \phi, \bar{\chi}) \ . \tag{2.28}$$

This implies that we may write $\omega_\pm$ as

$$\omega_+ = i\partial_\phi \bar{\partial}_\phi K - i\partial_\chi \bar{\partial}_\chi K, \qquad \omega_- = i\partial_\phi \bar{\partial}_\phi K + i\partial_\chi \bar{\partial}_\chi K \ . \tag{2.29}$$

Using the canonical form $J_\pm$ we define the metric g via

$$\omega_\pm = g J_\pm \ . \tag{2.30}$$

Finally, H is given by

$$H = -\bar{\partial}_\chi \partial_\phi \bar{\partial}_\phi K + \partial_\chi \partial_\phi \bar{\partial}_\phi K + \bar{\partial}_\phi \partial_\chi \bar{\partial}_\chi K - \partial_\phi \partial_\chi \bar{\partial}_\chi K \ . \tag{2.31}$$

Locally we have $H = dB$ and thus a possible representation of B is

$$B = \bar{\partial}_\phi \partial_\chi K - \bar{\partial}_\chi \partial_\phi K. \tag{2.32}$$

Hence, we have shown that all main objects are expressed locally in terms of second and third derivatives of a single real function K. The function K is well-defined modulo the addition of a function

$$\Lambda(\phi, \chi) + \bar{\Lambda}(\bar{\phi}, \bar{\chi}) + L(\phi, \bar{\chi}) + \bar{L}(\bar{\phi}, \chi) \ . \tag{2.33}$$

We would like to stress again that K enters linearily in all formulas and that this is an essential feature of the BiLP geometry.

3. $N = (2, 2)$ supersymmetric sigma-models

In this section we translate the properties of generalized Kähler geometry into sigma-model language. We start from the general $N = (1, 1)$ sigma-model written in terms of $N = (1, 1)$ superfields

$$S = \int_\Sigma d^2\sigma \, d^2\theta \ D_+ \Phi^\mu D_- \Phi^\nu E_{\mu\nu}(\Phi) \, , \tag{3.1}$$

where $E = \frac{1}{2}(g + B)$ with $H = dB$. Requiring the existence of an additional supersymmetry transformations of the form [3]

$$\delta_2(\epsilon) \Phi^\mu = \epsilon^+ D_+ \Phi^\nu J^\mu_{+\nu}(\Phi) + \epsilon^- D_- \Phi^\nu J^\mu_{-\nu}(\Phi) \, , \tag{3.2}$$

we find that the target space must have generalized Kähler geometry.

Next we introduce the $N = (2,2)$ superfields needed for a complete description of the $N = (2,2)$ sigma-model in $(2,2)$ superspace. We work in the coordinates adapted to the decomposition (2.7):

• $\ker(J_+ - J_-)$
These directions are parametrized by chiral ϕ and antichiral $\bar{\phi}$ fields defines as

$$\bar{\mathbb{D}}_\pm \phi = \mathbb{D}_\pm \bar{\phi} = 0 \ . \tag{3.3}$$

• $\ker(J_+ + J_-)$
These directions are parametrized by twisted chiral χ and twisted antichiral $\bar{\chi}$ fields defined as

$$\bar{\mathbb{D}}_+ \chi = \mathbb{D}_- \chi = \mathbb{D}_+ \bar{\chi} = \bar{\mathbb{D}}_- \bar{\chi} = 0 \ . \tag{3.4}$$

• $\text{coker}[J_+, J_-]$
These directions are parametrized by left semichiral $\mathbb{X}_L$ and left anti-semichiral fields $\bar{\mathbb{X}}_L$

$$\bar{\mathbb{D}}_+ \mathbb{X}_L = \mathbb{D}_+ \bar{\mathbb{X}}_L = 0 \,, \tag{3.5}$$

and right semichiral $\mathbb{X}_R$ and right anti-semichiral fields $\bar{\mathbb{X}}_R$

$$\bar{\mathbb{D}}_- \mathbb{X}_R = \mathbb{D}_- \bar{\mathbb{X}}_R = 0 \ . \tag{3.6}$$

Here $\mathbb{D}$ is the $N = (2,2)$ covariant derivative and we follow the notation in [12]. The chiral and twisted chiral superfields were studied in [3]. The semichiral superfields were introduced in [2]. In [12] we have proven that these superfields provide a full description of the general $N = (2,2)$ sigma-model. The most general sigma-model action is specified by a real function K

$$S = \int d^2\sigma d^4\theta \ K(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R) \ . \tag{3.7}$$

From a geometrical point of view, the function K is precisely the generalized Kähler potential that we discussed in the previous section.

The potential K is not uniquely defined: it can be shifted by the following combination

$$f(\phi, \chi, \mathbb{X}_L) + g(\phi, \bar{\chi}, \mathbb{X}_R) + \bar{f}(\bar{\phi}, \bar{\chi}, \bar{\mathbb{X}}_L) + \bar{g}(\bar{\phi}, \chi, \bar{\mathbb{X}}_R) \,, \tag{3.8}$$

which changes the action (3.7) at most by total derivatives. We refer to this shift as a "generalized Kähler gauge transformation". The geometrical interpretation of (3.8) was given in the previous section, see the discussion after (2.12). We identify $(\mathbb{X}_L, \bar{\mathbb{X}}_L)$ with the coordinates q and $(\mathbb{X}_R, \bar{\mathbb{X}}_R)$ with the coordinates P.

Furthermore, we can perform a Legendre transform along $\text{coker}([J_+, J_-])$, i.e., along the semichiral directions [4]. Starting from a parent action

$$\int d^2\sigma d^4\theta \ \left(K(\phi, \bar{\phi}, \chi, \bar{\chi}, U, \bar{U}, V, \bar{V}) - \mathbb{X}_L U - \bar{\mathbb{X}}_L \bar{U} - \mathbb{X}_R V - \bar{\mathbb{X}}_R \bar{V} \right) \,, \tag{3.9}$$

where U and V are unrestricted superfields, we may choose to integrate out semichiral superfields $\mathbb{X}_L$ and $\mathbb{X}_R$, which restricts $U = U_L$ to be left semichiral and $V = V_R$ to be right semichiral. The resulting semichiral action is $K(\phi, \bar{\phi}, \chi, \bar{\chi}, U_L, \bar{U}_L, V_R, \bar{V}_R)$. Integrating out U and V instead, we solve the system

$$\begin{aligned} \mathbb{X}_L = K_U \quad \bar{\mathbb{X}}_L = K_{\bar{U}} \\ \mathbb{X}_R = K_V \quad \bar{\mathbb{X}}_R = K_{\bar{V}} \,, \end{aligned} \tag{3.10}$$

to get $U = U(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$, *etc.* Substituting the solution into (3.9) yields the Legendre transformed action $\tilde{K}(\phi, \bar{\phi}, \chi, \bar{\chi}, \mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R)$. Similarily, integrating out $\mathbb{X}_L$ and V yields a third dual K and integrating out $\mathbb{X}_R$ and U yields a fourth. These symmetries are directly related to the fact that K has an interpretation as a generating function and therefore the Legendre transform corresponds to switching to different generating function as described in subsection 2.2.

To extract g and B from (3.7) we need to integrate out some of the Fermi-coordinates θ and get rid of auxiliary fields and thus arrive at the $N = (1,1)$ action (3.1). We introduce some notation which we use throughout the rest of the paper. The superfields carry indices of the following kind and range;

$$\begin{aligned} \phi^\alpha, \bar{\phi}^{\bar{\alpha}}\,, \quad \alpha = 1 \dots d_c \quad , \quad \chi^{\alpha'}, \bar{\chi}^{\bar{\alpha}'}\,, \quad \alpha' = 1 \dots d_t\,, \\ \mathbb{X}_L^a, \bar{\mathbb{X}}_L^{\bar{a}}\,, \quad a = 1 \dots d_s \quad , \quad \mathbb{X}_R^{a'}, \bar{\mathbb{X}}_R^{\bar{a}'}\,, \quad a' = 1 \dots d_s \,. \end{aligned} \tag{3.11}$$

We also use the collective notation $\mathcal{A} \equiv (\alpha, \bar{\alpha})$, $\mathcal{A}' \equiv (\alpha', \bar{\alpha}')$, $A \equiv (a, \bar{a})$ and $A' \equiv (a', \bar{a}')$. To reduce the $N = (2,2)$ action to its $N = (1,1)$ form, we introduce the $N = (1,1)$ superspace derivatives D and extra supercharges Q [3]:

$$D_\pm = \mathbb{D}_\pm + \bar{\mathbb{D}}_\pm\,, \qquad Q_\pm = i(\mathbb{D}_\pm - \bar{\mathbb{D}}_\pm)\,. \tag{3.12}$$

In terms of these, the (anti)chiral, twisted (anti)chiral and semi (anti)chiral superfields satisfy

$$\begin{aligned} Q_\pm\phi = J_c D_\pm \phi \quad , \quad Q_\pm \chi = \pm J_t D_\pm \chi\,, \\ Q_+ \mathbb{X}_L = J_s D_+ \mathbb{X}_L \quad , \quad Q_- \mathbb{X}_R = J_s D_- \mathbb{X}_R\,. \end{aligned} \tag{3.13}$$

For the pair (ϕ, χ) we use the same letters to denote the $N = (1,1)$ superfields, i.e., the lowest components of the $N = (2,2)$ superfields (ϕ, χ). Each of the semi (anti)chiral fields gives rise to two $N = (1,1)$ fields [2]:

$$\begin{aligned} X_L \equiv \mathbb{X}_L| \qquad \Psi_{L-} \equiv Q_- \mathbb{X}_L| \\ X_R \equiv \mathbb{X}_R| \qquad \Psi_{R+} \equiv Q_+ \mathbb{X}_R|\,, \end{aligned} \tag{3.14}$$

where a vertical bar means that we take the $\theta^2 \propto \theta - \bar{\theta}$ independent component. The conditions (3.13) then also imply

$$\begin{aligned} Q_+ \Psi_{L-} = J_s D_+ \Psi_{L-}\,, \qquad Q_- \Psi_{L-} = i\partial_= X_L \\ Q_- \Psi_{R+} = J_s D_- \Psi_{R+}\,, \qquad Q_+ \Psi_{R+} = i\partial_{+\!\!+} X_R\,. \end{aligned} \tag{3.15}$$

Using the relations (3.12)-(3.15) we reduce the $N=(2,2)$ action to its $N=(1,1)$ form according to:

$$\int d^2\sigma d^2\theta d^2\bar{\theta}\ K(\phi^{\mathcal{A}},\chi^{\mathcal{A}'},\mathbb{X}_L^A,\mathbb{X}_R^{A'})| = \int d^2\sigma \mathbb{D}^2\bar{\mathbb{D}}^2 K| = -\frac{i}{4}\int d^2\sigma D^2 Q_+ Q_- K|\ . \quad (3.16)$$

Provided that the matrix K_{LR} (2.15) is invertible, the auxiliary spinors Ψ_{L-}, Ψ_{R+} may be integrated out leaving us with a $N=(1,1)$ second order sigma-model action of the type originally discussed in [3]. From this the metric and antisymmetric B-field may be read off in terms of derivatives of K, and from the form of the second supersymmetry (3.13- 3.15) the complex structures $J_\pm$ are determined. We shall use a basis where the coordinates are arranged in a column as

$$\begin{pmatrix} X_L^A \\ X_R^{A'} \\ \phi^{\mathcal{A}} \\ \chi^{\mathcal{A}'} \end{pmatrix} . \quad (3.17)$$

when we compute the $N=(1,1)$ Lagrangian; the sum $E=\frac{1}{2}(g+B)$ of the metric g and B-field then takes on the explicit form (2.18) [17].

It is interesting that there are no corrections from chiral and twisted chiral fields in the semichiral sector (where the results agree with [2] and [11]), whereas in the chiral and twisted chiral sector the semichiral fields contribute substantially.

Thus, locally, all objects ($J_\pm$, g, B) are given in terms of second derivatives of a single real function K. By construction, the present geometry is generalized Kähler, and therefore satisfies all the relations from the previous section.

In addition we can add to (3.1) the potential term $\int d^2\sigma d^2\theta\ W(\Phi)$ and ask about the most general $N=(2,2)$ Landau-Ginzburg model. We present the detailed analysis in appendix A. The upshot is that the only allowed terms one can add to (3.7) are

$$\int d^2\sigma\ d\theta^+ d\theta^- \mathcal{W}(\phi) + \int d^2\sigma\ d\theta^+ d\bar{\theta}^- \tilde{\mathcal{W}}(\chi) + c.c\ . \quad (3.18)$$

and no potential is allowed in semichiral directions.

4. Linearization

As we have stressed a few times, the general expression (2.18) for E is nonlinear. However, a superficial look at (2.18) suggests that nonlinearity is of the quotient type. This is the main point of our paper: any generalized Kähler data comes from a quotient of a BiLP geometry with respect to a set of abelian isometries.

We start by presenting an $N=(2,2)$ sigma-model argument. At this level the quotient idea is almost obvious. Consider the action

$$\int d^2\sigma d^4\theta\ K(\phi,\bar{\phi},\chi,\bar{\chi},\phi_L+\chi_L,\bar{\phi}_L+\bar{\chi}_L,\phi_R+\bar{\chi}_R,\bar{\phi}_R+\chi_R)\,, \quad (4.1)$$

where we have taken the action (3.7) and replaced the semichiral entries by the combinations of new chiral and twisted chiral fields: ϕ_L, ϕ_R, χ_L and χ_R. Thus the theory (4.1) is defined over a manifold with a dimension $(\dim M + \dim(\mathrm{coker}[J_+, J_-]))$. Since the action (4.1) depends only on chiral and twisted chiral superfields, the target geometry is of the BiLP type. The action (4.1) has the following *complex* symmetries

$$\delta\phi_L = \lambda_L\,, \qquad \delta\chi_L = -\lambda_L\,, \qquad \delta\phi_R = \lambda_R\,, \qquad \delta\chi_R = -\bar{\lambda}_R\,. \tag{4.2}$$

The parameters satisfy

$$\bar{\mathbb{D}}_\pm\lambda_L = 0\,, \quad \mathbb{D}_-\lambda_L = 0\,, \quad \bar{\mathbb{D}}_\pm\lambda_R = 0\,, \quad \mathbb{D}_+\lambda_R = 0\,,$$

and thus they correspond to Kac-Moody symmetries, i.e., $\partial_{=}\lambda_L = 0$ and $\partial_{++}\lambda_R = 0$.

We can gauge these symmetries using semichiral fields as "connections". The action

$$\int d^2\sigma d^4\theta\; K(\phi,\bar{\phi},\chi,\bar{\chi},\phi_L+\chi_L+\mathbb{X}_L,\bar{\phi}_L+\bar{\chi}_L+\bar{\mathbb{X}}_L,\phi_R+\bar{\chi}_R+\mathbb{X}_R,\bar{\phi}_R+\chi_R+\bar{\mathbb{X}}_R) \tag{4.3}$$

is invariant under the gauge transformations

$$\delta\phi_L = \Lambda_L\,, \qquad \delta\chi_L = -\tilde{\Lambda}_L\,, \qquad \delta\mathbb{X}_L = -\Lambda_L + \tilde{\Lambda}_L\,, \tag{4.4}$$

$$\delta\phi_R = \Lambda_R\,, \qquad \delta\bar{\chi}_R = -\tilde{\Lambda}_R\,, \qquad \delta\mathbb{X}_R = -\Lambda_R + \tilde{\Lambda}_R\,, \tag{4.5}$$

(as well as their complex conjugates). The parameters $\Lambda_{L,R}$ are chiral whereas the parameters $\tilde{\Lambda}_{L,R}$ are twisted chiral. By fixing the gauge symmetry in (4.3),

$$\phi_L + \chi_L = 0\,, \qquad \phi_R + \bar{\chi}_R = 0\,, \tag{4.6}$$

we arrive at the action (3.7). Thus an $N = (2,2)$ sigma-model (3.7) on a generalized Kähler geometry can be thought of as a quotient of the sigma-model given by (4.1) defined on a certain BiLP geometry. In the rest of the paper we will translate this statement into geometrical terms. We refer to the target space BiLP geometry defined by (4.1) as an auxiliary local product space (ALP space).

5. Defining the ALPs

In this section we describe the ALPs in intrinsic geometrical terms. We are only concerned with the local properties of ALPs here, and our speculations about the global structure are presented in section 7. In particular we have to understand the extra requirements a BiLP geometry must satisfy for it to be an ALP, i.e., to have an generalized Kähler potential of the form in (4.1).

The generalized Kähler potential in (4.1) is invariant under the action of the following vector fields

$$k_L^a = \frac{\partial}{\partial\phi_L^a} - \frac{\partial}{\partial\chi_L^a}\,, \qquad k_R^{a'} = \frac{\partial}{\partial\phi_R^{a'}} - \frac{\partial}{\partial\bar{\chi}_R^{a'}}\,, \tag{5.1}$$

as well as their complex conjugates. The invariance of K with respect to these vectors implies that the metric g and H are invariant as well. Thus these vectors generate isometries. Moreover K obeys

$$K_{\phi_L^a \bar{\phi}_L^b} = K_{\chi_L^a \bar{\chi}_L^b} \,, \qquad K_{\phi_R^{a'} \bar{\phi}_R^{b'}} = K_{\chi_R^{a'} \bar{\chi}_R^{b'}} \;; \tag{5.2}$$

these relations imply that the metric g is indefinite. Thus k_L and k_R span two isotropic (null) subspaces with respect to the metric g defined in (2.30). However, for the original model (3.7) to make sense, we must require that

$$K_{\phi_L^a \bar{\phi}_R^{b'}} \tag{5.3}$$

is a nondegenerate matrix. Furthermore, the vectors $(\Pi k_L, \Pi k_R)$ are linearly independent of (k_L, k_R), where Π is the product structure of the BiLP geometry.

We now reformulate all these properties in a coordinate independent way. The ALP space is locally a trivial fibration over the generalized Kähler manifold M with fiber the vector space $\mathbb{C}^{2d_s}$, where $2d_s = \dim(\mathrm{coker}[J_+^M, J_-^M])$ and the complex vector fields k_L and k_R span $\mathbb{C}^{2d_s}$ (for clarity, we indicate quantities on the original manifold with a superscript M). Further, the vectors k_L and k_R in (5.1) correspond to the left and right Kac-Moody symmetries (4.2) of the σ-model, and, as observed in [16], satisfy

$$\nabla^{(+)} k_{LA} = 0 \,, \qquad \nabla^{(-)} k_{RA'} = 0 \,, \tag{5.4}$$

where the corresponding connections are defined in (2.3). We use the property (5.4) as the *definition* of left (right) Kac-Moody isometries of the geometry.

As every k_L is matched with a corresponding k_R, we identify the indices $A' \simeq A$ below. From (5.4) it follows that

$$\mathcal{L}_{k_L} g = 0 \,, \qquad \mathcal{L}_{k_R} g = 0 \,, \qquad \mathcal{L}_{k_L} H = 0 \,, \qquad \mathcal{L}_{k_R} H = 0 \;. \tag{5.5}$$

We also require that the isometries k_L and k_R respect the BiLP geometry; thus in addition to (5.5) we have the conditions

$$\mathcal{L}_{k_L} J_\pm = 0 \,, \qquad \mathcal{L}_{k_R} J_\pm = 0 \;. \tag{5.6}$$

Finally the conditions (5.2) become

$$k_{LA}^\mu g_{\mu\nu} k_{LB}^\nu = 0 \,, \qquad k_{RA}^\mu g_{\mu\nu} k_{RB}^\nu = 0 \,, \tag{5.7}$$

whereas $k_{LA}^\mu g_{\mu\nu} k_{RB}^\nu$ is required to be nondegenerate.

Definition 2. *We define the ALP geometry as a BiLP geometry*[4] *with an equal number of left and right Kac-Moody null abelian isometries (5.4) that respect the BiLP geometry and in addition satisfy (5.7). Furthermore, we require that the product structure* Π *maps the isometry directions into* $\mathrm{coker}[J_+^M, J_-^M]$.

[4] Recall that a BiLP geometry is a generalized Kähler geometry with the additional requirement $[J_+, J_-] = 0$, see the discussion in section 2.3.

Thus locally an ALP space has the structure $M \times (V_L \oplus V_R)$ in addition to the BiLP geometry. Here V_L (V_R) is an isotropic vector space with respect to the metric g_{ALP} and it is spanned by k_L (k_R) which are left (right) Kac-Moody isometries. The product structure Π maps $(V_L \oplus V_R)$ to the vectors tangential to $\text{coker}[J_+, J_-]$ in M.

It is a straightforward exercise to show that K in (4.1) satisfies this definitions. Indeed, this motivated the definition. However, the inverse statement that any ALP K can brought to the form (4.1) is less trivial and we collect the details of this proof in appendix B.

Theorem 3. *Locally, any generalized Kähler manifold M with its geometrical data can be thought of as a quotient of an ALP geometry by its Kac-Moody isometries.*

This is consistent with dimension of the ALP and the number of isometires:

$$\dim(ALP) = \dim M + \dim(\text{coker}[J_+, J_-]) \ .$$

This theorem explains the nonlinearities that appear in the metric and B field when they are expressed in terms of a generalized Kähler potential. The proof of this theorem is quite trivial at the level of sigma-model-see the previous section. However, the geometrical aspects of this theorem are less trivial, and the rest of the paper is devoted to clarifying these.

6. Quotients and sigma-models

In this section we review some old and discuss some new aspects of quotients. In particular, we are interested in the relation between the sigma-model approach to quotients and their geometrical interpretation. Our main goal is to explain how to perform the quotient with respect to Kac-Moody null isometries. Everywhere we will assume that the isometries commute.

We start by recalling some standard material on quotient metrics and sigma-models; for more details see [8]. We need this background to contrast it with the more exotic quotient constructions that follow. Consider a smooth manifold $\mathbb{M}$ and assume that a Lie group $\mathbf{G}$ acts smoothly on $\mathbb{M}$. If $\mathbf{G}$ acts freely and properly then the quotient space $\mathbb{M}/\mathbf{G}$ (i.e., the space of orbits) is a smooth manifold. There is a corresponding principal bundle

$$\begin{array}{ccc} \mathbb{M} & \longleftarrow & \mathbf{G} \\ {\scriptstyle p}\big\downarrow & & \\ \mathbb{M}/\mathbf{G} & & \end{array} \tag{6.1}$$

with p being a smooth projection. If $\mathbb{M}$ has a metric g and $\mathbf{G}$ acts as isometries, then we can define a metric $\tilde{g}$ on the quotient space. The corresponding nonzero vector fields k_A generated by $\mathbf{G}$ form a basis for a vertical subspace V_m of $T_m\mathbb{M}$, for $m \in \mathbb{M}$. Defining the horizontal subspace H_m as the set of vectors orthogonal to V_m we can define the metric $\tilde{g}$ on the quotient as

$$\tilde{g}(v, w) = g(\tilde{v}, \tilde{w}) \,, \tag{6.2}$$

where $v, w \in T_{p(m)}(\mathbb{M}/\mathbf{G})$ and their unique horizontal lift $\tilde{v}, \tilde{w} \in H_m \subset T_m\mathbb{M}$. Since $\mathbf{G}$ preserves the metric g, this definition of $\tilde{g}$ is independent of the choice of point m in the orbit $p^{-1}(p(m))$.

Alternatively, the choice of a horizontal subspaces can be described in terms of a choice of connection on the principal bundle (6.1). The orthogonal projection from $T_m\mathbb{M}$ to V_m defines a one-form θ with values in a Lie algebra. In the present context the connection is defined as

$$\theta_\mu^A = H^{AB} k_B^\nu g_{\nu\mu}, \tag{6.3}$$

where H^{AB} is the inverse of the matrix $H_{AB} \equiv k_A^\mu g_{\mu\nu} k_B^\nu$. Clearly,

$$\theta_\mu^A k_B^\mu = \delta^A{}_B. \tag{6.4}$$

Using the connection form θ in local coordinates, we get the expressions for $\tilde{g}$

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} - g_{\mu\rho} k_A^\rho \theta_\nu^A = g_{\mu\nu} - \theta_\mu^A k_A^\rho g_{\rho\nu} = g_{\mu\nu} - \theta_\mu^A H_{AB} \theta_\nu^B \ . \tag{6.5}$$

Clearly,

$$k_A^\mu \tilde{g}_{\mu\nu} = 0 \ . \tag{6.6}$$

The above geometrical picture for the quotient metric arises naturally in the sigma-model framework: Consider the bosonic sigma-model[5] defined over $\mathbb{M}$

$$S = \int d^2\sigma \ \partial_+ X^\mu g_{\mu\nu} \partial_- X^\nu \ . \tag{6.7}$$

Since k_A are the Killing vectors for g there is a corresponding global symmetry of the action S, given by

$$\delta X^\mu = \varepsilon^A k_A^\mu \ . \tag{6.8}$$

By introducing the world-sheet gauge fields A we promote this global symmetry to a gauge symmetry

$$S_{gauge} = \int d^2\sigma \ (\partial_+ X^\mu + k_A^\mu A_+^A) g_{\mu\nu} (\partial_- X^\nu + k_B^\nu A_-^B) \ . \tag{6.9}$$

Extremizing the action S_{gauge} with respect to A, we obtain a relation between the world-sheet and target space connections

$$A^A = -\theta_\mu^A dX^\nu = -X^*(\theta^A). \tag{6.10}$$

Thus the world-sheet gauge field A is just a pull-back of the connection θ by the map X. Reinserting this form of A in the action S_{gauge} yields the quotient sigma-model action, which is naturally defined over the space of orbits, $\mathbb{M}/\mathbf{G}$

$$S = \int d^2\sigma \ \partial_+ X^\mu \tilde{g}_{\mu\nu} \partial_- X^\nu \,, \tag{6.11}$$

where $\tilde{g}$ is defined in (6.5). The sigma-model derivation of the quotient metric naturally produces the connection form θ.

[5]The supersymmetric $N = (1, 1)$ sigma-models are treated in an identical fashion.

JHEP04(2007)061

We now consider more exotic quotients. In particular, we assume that $\mathbb{M}$ admits an invariant metric g and an invariant two-form[6], i.e.,

$$\mathcal{L}_{k_A} g = 0\,, \qquad \mathcal{L}_{k_A} B = 0. \tag{6.12}$$

The corresponding bosonic sigma-model on $\mathbb{M}$

$$S = \int d^2\sigma\ \partial_+ X^\mu E_{\mu\nu} \partial_- X^\nu \tag{6.13}$$

with $E = \frac{1}{2}(g + B)$ is invariant under a global symmetry (6.8). Gauging this symmetry exactly in the same fashion as before and extremizing the gauge action we find

$$\begin{aligned} A_+^A &= -(\theta_L)_\mu^A \partial_+ X^\mu \\ A_-^A &= -(\theta_R)_\mu^A \partial_- X^\mu\,, \end{aligned} \tag{6.14}$$

with the target space connections are defined as

$$(\theta_L)_\mu^A = E_{\mu\nu} k_B^\nu \mathcal{H}^{BA}, \qquad (\theta_R)_\mu^A = \mathcal{H}^{AB} k_B^\nu E_{\nu\mu}\,, \tag{6.15}$$

where $\mathcal{H}^{AB}$ is the inverse of $\mathcal{H}_{AB} := k_A^\mu E_{\mu\nu} k_B^\nu$. Now we have two different connection forms θ_L and θ_R, left and right, which both satisfy

$$(\theta_L)_\mu^A k_B^\mu = \delta^A{}_B, \qquad (\theta_R)_\mu^A k_B^\mu = \delta^A{}_B. \tag{6.16}$$

Plugging (6.14) into the gauged action produces the quotient $\tilde{E}$

$$\tilde{E}_{\mu\nu} = E_{\mu\nu} - (\theta_L)_\mu^A k_A^\rho E_{\rho\nu} = E_{\mu\nu} - E_{\mu\rho} k_A^\rho (\theta_R)_\nu^A = E_{\mu\nu} - (\theta_L)_\mu^A \mathcal{H}_{AB} (\theta_R)_\nu^B \tag{6.17}$$

which satisfies

$$k_A^\mu \tilde{E}_{\mu\nu} = 0\,, \qquad \tilde{E}_{\mu\nu} k_A^\nu = 0 \tag{6.18}$$

The quotient metric and B-field are given by the symmetric and antisymmetric part of $\tilde{E}$, respectively. Thus we see that the sigma-model offers a different quotient which involves the choice of two different connections, i.e., different choice of left and right horizontal spaces.

Finally, we discuss an even more exotic quotient involving null Kac-Moody isometries. This is the quotient that we actually use to descend from the ALP space to the underlying generalized Kähler geometry. We derive this quotient using a sigma-model construction, and then describe it geometrically.

Consider a manifold $\mathbb{M}$ that admits two sets of null abelian Kac-Moody isometries generated by left and right vector fields k_{LA} and k_{RA}, respectively. Assume that these vector fields satisfy the left and right Kac-Moody condition

$$\nabla^{(+)} k_{LA} = 0\,, \qquad \nabla^{(-)} k_{RA} = 0\,, \tag{6.19}$$

[6]This property can be relaxed and one can require that the closed three-from $H = dB$ is invariant [9]. However, we do not discuss this general case here.

and moreover that they are null

$$k^\mu_{LA} g_{\mu\nu} k^\nu_{LA} = 0\,, \qquad\qquad k^\mu_{RA} g_{\mu\nu} k^\nu_{RA} = 0\ . \tag{6.20}$$

The conditions (6.19) imply invariance of the metric as well as the conditions

$$k^\mu_{LA} H_{\mu\nu\rho} = \partial_{[\nu}\alpha^{LA}_{\rho]}\,, \qquad k^\mu_{RA} H_{\mu\nu\rho} = -\partial_{[\nu}\alpha^{RA}_{\rho]} \tag{6.21}$$

with $\alpha_\mu = g_{\mu\nu}k^\nu$.

Now consider a sigma-model on $\mathbb{M}$ with the standard action (6.13), which is chosen to be invariant under the Kac-Moody symmetries

$$\delta X^\mu = \varepsilon^A_L(z) k^\mu_{LA}(X) + \varepsilon^A_R(\bar z) k^\mu_{RA}(X)\ . \tag{6.22}$$

Inspired by [9], we gauge this symmetry (promoting $\varepsilon_L(z) \to \lambda_L(z,\bar z),\ \varepsilon_R(\bar z) \to \lambda_R(z,\bar z)$) and obtain the gauged action

$$S_{gauge} = \int d^2\sigma\, (\partial_+ X^\mu + k^\mu_{RA}\tilde A^A_+) g_{\mu\nu} (\partial_- X^\nu + k^\nu_{LB} A^B_-) - \frac{1}{2}\partial_+ X^\mu\,(g_{\mu\nu} - B_{\mu\nu})\,\partial_- X^\nu\,, \tag{6.23}$$

with

$$\delta A = -d\lambda_L\,, \qquad \delta\tilde A = -d\lambda_R\ . \tag{6.24}$$

Note that only half of the gauge fields appear in the gauged action (6.23): The fields $A_+, \tilde A_-$ does not appear, but nevertheless the action is fully gauge invariant. The various terms in the action are not gauge invariant by themselves but their variations cancel (after integration by parts). There is also a term from the variation of the first term that is linear in gauge fields, which of course could never be cancelled by the second term. Fortunately it is zero using the null condition (6.20).

Extremizing this action gives

$$\begin{aligned} \tilde A^A_+ &= -(\theta_R)^A_\mu \partial_+ X^\mu \\ A^A_- &= -(\theta_L)^A_\mu \partial_- X^\mu\,, \end{aligned} \tag{6.25}$$

with the target space connections defined as

$$(\theta_R)^A_\mu = g_{\mu\nu} k^\nu_{LB} h^{BA}, \qquad\qquad (\theta_L)^A_\mu = h^{AB} k^\nu_{RB} g_{\nu\mu}\,, \tag{6.26}$$

where h^{AB} is the inverse of

$$h_{AB} = k^\mu_{RA} g_{\mu\nu} k^\nu_{LB}\,, \tag{6.27}$$

and must be nondegenerate for the construction to work. The connections (6.26) satisfy the following properties

$$(\theta_L)^A_\mu k^\mu_{LB} = \delta^A{}_B, \qquad (\theta_R)^A_\mu k^\mu_{RB} = \delta^A{}_B, \qquad (\theta_L)^A_\mu k^\mu_{RB} = 0\,, \qquad (\theta_R)^A_\mu k^\mu_{LB} = 0. \tag{6.28}$$

Finally the quotient sigma model gives rise to $\tilde E$

$$\tilde E_{\mu\nu} = E_{\mu\nu} - E_{\mu\rho} k^\rho_{LA} (\theta_L)^A_\nu = E_{\mu\nu} - (\theta_R)^A_\mu k^\rho_{RA} E_{\rho\nu} = E_{\mu\nu} - (\theta_R)^A_\mu h_{AB} (\theta_L)^B_\nu\ . \tag{6.29}$$

This gives $\tilde{g}$ which satisfies

$$k^{\mu}_{RA}\tilde{g}_{\mu\nu} = 0\,, \qquad \tilde{g}_{\mu\nu}k^{\nu}_{LA} = 0\,, \qquad k^{\mu}_{LA}\tilde{g}_{\mu\nu} = 0\,, \qquad \tilde{g}_{\mu\nu}k^{\nu}_{RA} = 0 \tag{6.30}$$

and thus it is a well-defined tensor on the quotient. The $\tilde{B}$ we get from this $\tilde{E}$ is not zero when contracted with any killing vector. However, when we compute $\tilde{H}$ from $\tilde{B}$ using the identities

$$k^{\mu}_{LA}k^{\nu}_{LB}H_{\mu\nu\rho} = 0\,, \quad k^{\mu}_{RA}k^{\nu}_{RB}H_{\mu\nu\rho} = 0\,, \quad k^{\mu}_{LA}k^{\nu}_{RB}H_{\mu\nu\rho} = \frac{1}{2}\partial_{\rho}h_{BA}\,, \tag{6.31}$$

we get

$$\tilde{H}_{\mu\nu\rho} = H_{\mu\nu\rho} - k^{\alpha}_{LA}(\theta_L)^{A}_{[\mu}H_{\nu\rho]\alpha} - k^{\alpha}_{RA}(\theta_R)^{A}_{[\mu}H_{\nu\rho]\alpha} + k^{\alpha}_{LA}k^{\beta}_{RB}(\theta_L)^{A}_{[\mu}(\theta_R)^{B}_{\nu}H_{\rho]\alpha\beta}\,, \tag{6.32}$$

which obeys

$$\tilde{H}_{\mu\nu\rho}k^{\rho}_{LA} = \tilde{H}_{\mu\nu\rho}k^{\rho}_{RA} = 0\ . \tag{6.33}$$

This is the geometric form of the quotient that allows us to descend from the ALPs to produce a generalized Kähler manifold.

7. Global Issues

Hitherto, we have discussed only the local geometry of the ALP space. *A priori*, it is not clear if our construction makes sense globally, at least in the present form. We would like to make a few speculative remarks about the possibility of a global interpretation.

Consider diffeomorphisms along the leaves of the Poisson structure σ that preserve our special coordinates. In the sigma-model this corresponds to the following

$$\int d^2\sigma d^4\theta\ K\left(\phi,\bar{\phi},\chi,\bar{\chi},f_L(\mathbb{X}_L),\bar{f}_L(\bar{\mathbb{X}}_L),f_R(\mathbb{X}_R),\bar{f}_R(\bar{\mathbb{X}}_R)\right)\,, \tag{7.1}$$

where f_L and f_R are arbitrary functions. Correspondingly, for the ALP sigma-model we have

$$\int d^2\sigma d^4\theta\ K\left(\phi,\bar{\phi},\chi,\bar{\chi},f_L(\phi_L+\chi_L),\bar{f}_L(\bar{\phi}_L+\bar{\chi}_L),f_R(\phi_R+\bar{\chi}_R),\bar{f}_R(\bar{\phi}_R+\chi_R)\right)\ . \tag{7.2}$$

We can study how the geometric data of the ALP transforms under such a transformation, that is, how $E = \frac{1}{2}(g+B)$ transforms; we find

$$E\ \longrightarrow M^t E M,$$

where

$$M = diag(1,1,1,1,\frac{\partial f_L}{\partial X_L},\frac{\partial \bar{f}_L}{\partial \bar{X}_L},\frac{\partial f_R}{\partial X_R},\frac{\partial \bar{f}_R}{\partial \bar{X}_R},\frac{\partial f_L}{\partial X_L},\frac{\partial \bar{f}_L}{\partial \bar{X}_L},\frac{\partial f_R}{\partial X_R},\frac{\partial \bar{f}_R}{\partial \bar{X}_R}),$$

and we assumed that E is ordered as in (C.2). Thus we see that the diffeomorphisms along coker$[J_+,J_-]$ induce transformations of the fiber as for a vector bundle. This suggests that ALPs can be thought of as a subbundle[7] of TM such that the fibers lie along the leaves of σ.

[7] For this to make sense, we need to assume that the Poisson structure σ is regular.

JHEP04(2007)061

8. Summary

In this work we have shown that, locally, for any generalized Kähler manifold there exits an auxiliary space (ALP), $M \times \mathbb{R}^{2d_s}$ with a particular simple complex geometry and such that M is a specific quotient of this auxiliary space. This explains the nonlinearities in the metric with respect to generalized Kähler potential. Our construction is based on a simple sigma-model argument which, we hope, clarifies the nature of the semichiral fields.

On the mathematical side it would be nice to define the ALP geometry intrinsically in geometrical terms, without any reference to a generalized Kähler potential. If it is possible, then we will have an alternative derivation of the existence of a generalized Kähler potential. Another interesting problem is to see if the ALPs can be defined globally as a vector bundle, at least for regular generalized Kähler manifolds.

For physics the ALP construction offers the opportunity to solve some problems without using semichiral fields. For example, the problem of a topological twist of $N = (2,2)$ sigma-model involves many auxiliary fields with the main complication coming from semichiral sector. The chiral and the twisted chiral sectors are relatively simple when it comes to the topological twist. This suggests that the twist should be performed in the ALPs and the quotient constructed afterwards.

In [10] it is shown how to construct hyperkähler metrics using semichiral but no (twisted) chiral fields. These models should provide interesting applications of our linearization.

Also the quotients and dualities of generalized Kähler manifold can be analyzed in the ALP picture. Any isometry of M can be lifted to an isometry of the corresponding ALP which commutes with the Kac-Moody isometries. We plan to come back to these issues elsewhere.

Acknowledgments

We are grateful to the 2006 Simons Workshop for providing the stimulating atmosphere where this work was initiated. We thank M.T. Grisaru for discussions. The work of U.L. is supported by EU grant (Superstring theory) MRTN-2004-512194 and VR grant 621-2003-3454. The work of MR was supported in part by NSF grant no. PHY-0354776 and Supplement for International Cooperation with Central and Eastern Euorpe PHY 0300634. The research of R.v.U. was supported by Czech ministry of education contract No. MSM0021622409 and by Kontakt grant ME649. The research of M.Z. was supported by VR-grant 621-2004-3177.

A. The general $N = (2,2)$ Landau-Ginzburg models

The $N = (1,1)$ Landau-Ginzburg model is given by the following action

$$S = \int d^2\sigma\, d^2\theta\ [D_+\Phi^\mu D_-\Phi^\nu E_{\mu\nu}(\Phi) + W(\Phi)],$$

where $E = \frac{1}{2}(g + B)$ with $H = dB$ and W is an arbitrary real function. This action gives rise to the equation of motion

$$D_+ D_- \Phi^\lambda + \Gamma^{-\lambda}_{\sigma\nu} D_+ \Phi^\sigma D_- \Phi^\nu - \frac{1}{2} g^{\lambda\nu} \partial_\nu W = 0.$$

We now consider the restrictions on W that follow from imposing invariance under additional supersymmetry transformations of the form [3]

$$\delta_2(\epsilon)\Phi^\mu = \epsilon^+ D_+ \Phi^\nu J^\mu_{+\nu}(\Phi) + \epsilon^- D_- \Phi^\nu J^\mu_{-\nu}(\Phi) \ .$$

The kinetic and potential terms are invariant independently if the following conditions are satisfied

$$J^t_\pm g = -g J_\pm, \qquad \nabla^{(\pm)} J_\pm = 0\,, \qquad J^\mu_{+\nu} \partial_\mu W = \partial_\nu W_\pm,$$

where $W_\pm$ are some functions. We also impose the on-shell supersymmetry algebra. The commutator of two second supersymmetry transformations is

$$\begin{aligned}
[\delta_2(\epsilon_1), \delta_2(\epsilon_2)]\Phi^\mu =& 2i\epsilon_1^+ \epsilon_2^+ \partial_{\mathbin{+\mkern-10mu+}} \Phi^\lambda (J^\mu_{+\nu} J^\nu_{+\lambda}) + 2i\epsilon_1^- \epsilon_2^- \partial_= \Phi^\lambda (J^\mu_{-\nu} J^\nu_{-\lambda}) \\
&- \epsilon_1^+ \epsilon_2^+ D_+ \Phi^\lambda D_+ \Phi^\rho \mathcal{N}^\mu{}_{\lambda\rho}(J_+) - \epsilon_1^- \epsilon_2^- D_- \Phi^\lambda D_- \Phi^\rho \mathcal{N}^\mu{}_{\lambda\rho}(J_-) \\
&+ (\epsilon_1^+ \epsilon_2^- + \epsilon_1^- \epsilon_2^+)(J^\mu_{+\nu} J^\nu_{-\lambda} - J^\mu_{-\nu} J^\nu_{+\lambda})(D_+ D_- \Phi^\lambda + \Gamma^{-\lambda}_{\sigma\nu} D_+ \Phi^\sigma D_- \Phi^\nu) \ .
\end{aligned}$$

which should be

$$[\delta_2(\epsilon_1), \delta_2(\epsilon_2)]\Phi^\mu = -2i\epsilon_1^+ \epsilon_2^+ \partial_{\mathbin{+\mkern-10mu+}} \Phi^\mu - 2i\epsilon_1^- \epsilon_2^- \partial_= \Phi^\mu \ .$$

Thus we find that $J_\pm$ are complex structures. However using the Landau-Ginzburg equations of motion the last term in the algebra can be canceled only upon the additional requirement that

$$\sigma^{\mu\nu} \partial_\nu W = 0,$$

where $\sigma = [J_+, J_-] g^{-1}$. Therefore we conclude that W should be a Casimir function[8] for the Poisson structure σ and moreover that W is the real part of a function holomorphic with respect to both J_+ and J_-.

In $N = (2, 2)$ language this implies that the only potential terms possible are

$$\mathrm{Re} \int d^2\sigma \, d\theta^+ d\theta^- \mathcal{W}(\phi) + \mathrm{Re} \int d^2\sigma \, d\theta^+ d\bar{\theta}^- \tilde{\mathcal{W}}(\chi)\,,$$

No potential is possible along semichiral directions, i.e., along coker$[J_+, J_-]$. Indeed, due to the specific nature of semichiral fields, no potential term can be written for them in $N = (2, 2)$ language.

[8]The Casimir function of σ has vanishing Poisson bracket with any function from $C^\infty(M)$.

B. ALP generalized Kähler potential

In this appendix, we show that it is possible to start from the definition of the ALP space (see section 5) and choose a generalized Kähler potential on the ALPs that is invariant with respect to its Kac-Moody isometries; this guarantees that it is possible to descend to the underlying generalized Kähler geometry.

By definition, an ALP space is $M \times (V_L \oplus V_R)$ with BiLP geometry, where the vector spaces $V_{L,R}$ are isotropic (null) vector spaces with respect to the metric on the ALP space; they are spanned by $k_{L,R}$, the left and right commuting Kac-Moody isometries (in the sense of (5.4)), respectively. The product structure Π of the BiLP maps the Killing vectors $k_{L,R}$ to the vectors $\Pi k_{L,R}$ that span the directions tangential to coker$[J_+^M, J_-^M]$. Because the isometries generated by $k_{L,R}$ are Kac-Moody, they are *complex*. As the isometries respect the BiLP data, we can choose a local coordinates adapted both to the BiLP geometry (see subsection 2.3) and to the isometries:

$$k_L^a = \frac{\partial}{\partial \phi_L^a} - \frac{\partial}{\partial \chi_L^a}\,, \qquad \bar{k}_L^a = \frac{\partial}{\partial \bar{\phi}_L^a} - \frac{\partial}{\partial \bar{\chi}_L^a}\,,$$

and the right isometries as

$$k_R^a = \frac{\partial}{\partial \phi_R^{a'}} - \frac{\partial}{\partial \bar{\chi}_R^{a'}}\,, \qquad \bar{k}_R^a = \frac{\partial}{\partial \bar{\phi}_R^{a'}} - \frac{\partial}{\partial \chi_R^{a'}}\,.$$

In these coordinates, the invariance of the generalized Kähler potential implies that it is independent of $\phi_L^a - \chi_L^a, \phi_R^{a'} - \bar{\chi}_R^{a'}$, *etc.* However, we shall not use this explicit coordinate dependent form.

Before we prove the main result, we explain the problem by considering the analogous issue for holomorphic isometries of a Kähler manifold. Consider a number of commuting Killing vectors $k^a = k^a(z)\partial + \bar{k}^a(\bar{z})\bar{\partial}$; these generate an isometry if they preserve the Kähler potential up to the real part of holomorphic functions f_a:

$$k^a K = f_a + \bar{f}_a\,. \qquad \text{(B.1)}$$

As these vectors commute, we have

$$0 = (k^a k^b - k^b k^a)K = k^a f_b + k^a \bar{f}_b - k^b f_a + k^b \bar{f}_a\,. \qquad \text{(B.2)}$$

As the functions f_a are holomorphic, this implies

$$k^a f_b - k^b f_a = ic_{ab}\,, \qquad k^a \bar{f}_b - k^b \bar{f}_a = -ic_{ab}\,, \qquad \text{(B.3)}$$

where c_{ab} are real constants. These constants are an obstruction to the existence of an invariant Kähler potential-no shift of K by the real part of a holomorphic function can eliminate all the f_a's.

We now show that *no* such obstruction exists for the (null) Kac-Moody isometries of an ALP space. We first consider the left sector only; the right sector is completely independent, and the same argument applies to it. The condition for invariance of the geometry implies the following condition on the generalized Kähler potential:

$$k_L^a K = f_a(\phi,\chi) + g_a(\phi,\bar{\chi}) + l_a(\bar{\phi},\chi) \; ; \tag{B.4}$$

This equation, the analog of (B.1), follows from imposing invariance of the action (4.1) under the symmetries (4.2); one may also deduce this directly from the condition (5.4). Because the k_L^a are null and span an isotropic (null) space,

$$g_{\mu\nu} k_a^\mu \bar{k}_b^\nu = 0\,,$$

where for brevity we write $k_a \equiv k_L^a$ throughout the rest of this appendix. Because of the BiLP structure of the ALP, this can be rewritten as:

$$[k_a(\Pi\bar{k}_b) + (\Pi k_a)\bar{k}_b]K = 0\,,$$

where Π is the local product structure of the BiLP. Because the vector fields k_a commute and preserve Π, using (B.4) we find

$$(\Pi k_a)(\bar{f}_b + \bar{g}_b + \bar{l}_b) + (\Pi\bar{k}_b)(f_a + g_a + l_a) = 0\ .$$

The dependence of f_a, g_a, l_a on the coordinates implies

$$(\Pi k_a)\bar{f}_b = 0\,, \qquad (\Pi k_a)\bar{g}_b = -k_a\bar{g}_b\,, \qquad (\Pi k_a)\bar{l}_b = k_a\bar{l}_b\,,$$

and hence we find that the null condition becomes

$$\mathrm{Re}(k_a\bar{l}_b - \bar{k}_a g_b) = 0\ . \tag{B.5}$$

Next we consider the condition that the Kac-Moody isometries commute; this gives

$$0 = (k_a\bar{k}_b - \bar{k}_b k_a)K = 2i\,\mathrm{Im}(k_a\bar{l}_b - \bar{k}_a g_b)\,,$$

and thus we conclude that

$$k_a\bar{l}_b - \bar{k}_b g_a = 0\ . \tag{B.6}$$

In any contractible patch, this implies

$$g_a = k_a L(\phi,\bar{\chi}) + g_a^0(\phi)\,, \qquad l_a = k_a\bar{L}(\bar{\phi},\chi) + l_a^0(\chi)\ . \tag{B.7}$$

If we substitute this into (B.4), we see that by shifting the generalized Kähler potential

$$K \rightarrow K - (L + \bar{L})\,,$$

we obtain a simple holomorphic transformation:

$$k_a K = f_a(\phi,\chi) + g_a^0(\phi) + l_a^0(\chi) \equiv \tilde{f}_a(\phi,\chi)\ .$$

Now considering the commutator

$$0 = (k_a k_b - k_b k_a)K = k_a\tilde{f}_b - k_b\tilde{f}_a\ ;$$

we find

$$\tilde{f}_a = k_a\Lambda(\phi,\chi)\ .$$

Shifting the generalized Kähler potential by Λ allows us to eliminate $\tilde{f}_a$, and hence find an invariant K. We can proceed analogously for the right Kac-Moody isometries. The reason that we found no possible obstruction is that (B.6), in contrast to the Kähler case (B.2), has only two terms.

C. The descent from the ALPs

In this appendix, we give details of the computations involved in performing the quotient (6.29) that takes us from the ALP space down to the generalized Kähler geometry. We evaluate the general formulas of section 6 for the special case when the B-field itself is preserved by the Kac-Moody symmetries; in fact we have

$$k^{\mu}_{LA} E_{\mu\nu} = E_{\mu\nu} k^{\nu}_{RA} = 0 \; . \tag{C.1}$$

We first consider the case without chiral or twisted chiral fields, that is, $\ker[J_+, J_-] = \emptyset$ and the Poisson structure σ is nondegenerate, and hence its inverse is a symplectic form.

C.1 Maximally symplectic case

Here we concentrate on the case when the whole manifold is $\text{coker}[J_+, J_-]$. An example of this situation is given by a hyperkähler manifold with J_+ and J_- two different non-commuting complex structures. Thus locally any d-dimensional hyperkähler metric can be thought of a quotient metric of a $2d$-dimensional ALP space.

We start from the reduction of the $N = (2, 2)$ action

$$K\left(\mathbb{X}_L, \bar{\mathbb{X}}_L, \mathbb{X}_R, \bar{\mathbb{X}}_R\right) \tag{C.2}$$

to $N = (1, 1)$ components

$$\begin{pmatrix} D_+X_L & \Psi_{L+} & D_+X_R & \Psi_{R+} \end{pmatrix} E \begin{pmatrix} D_-X_L \\ \Psi_{L-} \\ D_-X_R \\ \Psi_{R-} \end{pmatrix} \tag{C.3}$$

with the auxiliary fields remaining. E is given by the following formula

$$E = \begin{pmatrix} 0 & K_{LL} + JK_{LL}J & JK_{LR}J & 0 \\ 0 & 0 & 0 & 0 \\ 0 & K_{RL} & 0 & 0 \\ K_{RL} & JK_{RL}J & K_{RR} + JK_{RR}J & 0 \end{pmatrix} \tag{C.4}$$

Notice that there is actually no Ψ_{L+} or Ψ_{R-} in the action (just as half the the gauge fields dropped out in (6.23)).

To compare to the ALP we introduce

$$\Psi_{L\pm} = JD_{\pm}\Lambda_L \, , \qquad \Psi_{R\pm} = JD_{\pm}\Lambda_R \; . \tag{C.5}$$

It is sometimes convenient to rearrange rows and columns accordingly

$$\begin{pmatrix} X_L & X_R & \Lambda_L & \Lambda_R \end{pmatrix} \tag{C.6}$$

with the result that (C.4) becomes

$$E = \begin{pmatrix} 0 & JK_{LR}J & K_{LL} + JK_{LL}J & 0 \\ 0 & 0 & K_{RL} & 0 \\ 0 & 0 & 0 & 0 \\ K_{RL} & K_{RR} + JK_{RR}J & JK_{RL}J & 0 \end{pmatrix} . \tag{C.7}$$

Starting instead from the ALP we replace the semichiral fields by combinations of chiral and twisted chiral fields according to

$$K(\phi_L + \chi_L, \bar{\phi}_L + \bar{\chi}_L, \phi_R + \bar{\chi}_R, \bar{\phi}_R + \chi_R). \tag{C.8}$$

When we go to $N = (1,1)$ components we treat L-fields and R-fields differently in terms of integration by parts to mimic the way the semichiral fields are treated. With R-fields we integrate by parts with D_+ and with L-fields we integrate by parts with D_-. Using a notation where the K_{ab}, $K_{ab'}$ *etc.* entries are suppressed and only the overall coefficients are written, the result is a matrix with rows and columns according to

$$\left(\begin{array}{cccc|cccc} \phi_L & \bar{\phi}_L & \phi_R & \bar{\phi}_R & \chi_L & \bar{\chi}_L & \chi_R & \bar{\chi}_R \end{array} \right) \tag{C.9}$$

as

$$\left(\begin{array}{cccc|cccc} 0 & 2 & -1 & 1 & 0 & -2 & 1 & -1 \\ 2 & 0 & 1 & -1 & -2 & 0 & -1 & 1 \\ 1 & 3 & 0 & 2 & 1 & -1 & 2 & 0 \\ 3 & 1 & 2 & 0 & -1 & 1 & 0 & 2 \\ \hline 0 & 2 & -1 & 1 & 0 & -2 & 1 & -1 \\ 2 & 0 & 1 & -1 & -2 & 0 & -1 & 1 \\ -1 & 1 & -2 & 0 & -1 & -3 & 0 & -2 \\ 1 & -1 & 0 & -2 & -3 & -1 & -2 & 0 \end{array} \right) ; \tag{C.10}$$

thus the coefficient of $K_{a_L \bar{b}_L}$ is 2, the coefficient of $K_{a_L b_R}$ is -1, *etc.*

To go to the form we get from the semichiral reduction we identify

$$X_L = \phi_L + \chi_L, \tag{C.11}$$

$$X_R = \phi_R + \bar{\chi}_R. \tag{C.12}$$

Since $\Psi_{L-} = Q_- \mathbb{X}_L$ we may identifiy

$$\Psi_{L-} = Q_-(\phi_L + \chi_L) = JD_-(\phi_L - \chi_L) \tag{C.13}$$

and similarly

$$\Psi_{+R} = JD_+(\phi_R - \bar{\chi}_R) \tag{C.14}$$

which leads to the definition

$$\Lambda_L = \phi_L - \chi_L \tag{C.15}$$

$$\Lambda_R = \phi_R - \bar{\chi}_R \tag{C.16}$$

Rearranging rows and columns of the matrix (C.10) above accordingly, we get

$$\left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & 1 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & -1 & 2 & 0 & 0 & 0 \end{array}\right) \tag{C.17}$$

which should be compared to the ALP's E, (C.4) with the entries evaluated.

Finally let us write the quotient formula (6.29) explicitly in these adapted coordinates. Using obvious matrix notation we can write the isometry vectors as follows

$$k_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \qquad k_L = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{C.18}$$

and the connections are

$$\theta_L = \Big(J_s K_{LR}^{-1} J_s K_{RL} \;\; J_s K_{LR}^{-1} C_{RR} \;\; 1 \;\; 0 \Big), \qquad \theta_R^t = \begin{pmatrix} -C_{LL} K_{LR}^{-1} J_s \\ K_{RL} J_s K_{LR}^{-1} J_s \\ 0 \\ 1 \end{pmatrix}, \tag{C.19}$$

which satisfy $\theta_R k_R = 1$ and $k_L^t \theta_L^t = 1$. Using $\tilde{E} = E - \theta_R^t (k_R^t E k_L) \theta_L$ we get

$$\tilde{E} = \begin{pmatrix} C_{LL} K_{LR}^{-1} J_s K_{RL} & J_s K_{LR} J_s + C_{LL} K_{LR}^{-1} C_{RR} & 0 & 0 \\ -K_{RL} J_s K_{LR}^{-1} J_s K_{RL} & -K_{RL} J_s K_{LR}^{-1} C_{RR} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{C.20}$$

which is exactly the E in (2.18) along coker$[J_+, J_-]$.

C.2 The general case

We briefly discuss the general case; because they are so long, we omit the explicit version of some formulae.

Including tourists in the ALP (i.e., other fields $\phi^{\mathcal{A}}, \chi^{\mathcal{A}'}$ which do not belong to coker$[J_+, J_-]$), the Kähler potential depends on

$$\phi^{\mathcal{A}}, \chi^{\mathcal{A}'}, \phi_L^A, \phi_R^{A'}, \chi_L^A, \chi_R^{A'} \; . \tag{C.21}$$

Going down to $N = (1,1)$ we find the metric and B-field. Remembering to treat the left fields ϕ_L, χ_L and right fields ϕ_R, χ_R differently with respect to the partial integration we find an E which, with rows and columns labelled according to

$$\Big(\phi^{\mathcal{A}}, \chi^{\mathcal{A}'}, 2(\phi_L + \chi_L)^A, 2(\phi_R + \bar{\chi}_R)^{A'}, 2(\phi_L - \chi_L)^A, 2(\phi_R - \bar{\chi}_R)^{A'}\Big) \,, \tag{C.22}$$

can be written as

$$\begin{pmatrix} 0 & 2 & 0 & -2 & 0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 0 & 0 & -1 & 1 & 0 & 2 & 0 & 0 \\ 2 & 0 & -2 & 0 & 0 & 0 & 1 & -1 & 2 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 1 & 1 & 0 & 2 & -1 & 1 & 0 & 0 \\ 2 & 0 & 0 & 2 & 1 & 1 & 2 & 0 & 1 & -1 & 0 & 0 \end{pmatrix} \tag{C.23}$$

From this we read off h_{AB} (defined in (6.27));

$$h_{AB} = JK_{RL}J \; . \tag{C.24}$$

Here we use the matrix notation for the isometry vectors:

$$k_R = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} , \qquad k_L = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} . \tag{C.25}$$

The connections are

$$\theta_R = \begin{pmatrix} -C_{cL}K_{LR}^{-1}J \\ -C_{tL}K_{LR}^{-1}J \\ -C_{LL}K_{LR}^{-1}J \\ K_{RL}JK_{LR}^{-1}J \\ 0 \\ 1 \end{pmatrix} , \qquad \theta_L = \begin{pmatrix} JK_{LR}^{-1}C_{Rc} \\ JK_{LR}^{-1}A_{Rt} \\ JK_{LR}^{-1}JK_{RL} \\ JK_{LR}^{-1}C_{RR} \\ 1 \\ 0 \end{pmatrix} , \tag{C.26}$$

which when combined with (6.29) gives the correct full $\tilde{E}$ on the quotient space (2.18).

References

[1] A. Bredthauer, U. Lindström, J. Persson and M. Zabzine, *Generalized Kähler geometry from supersymmetric sigma models*, *Lett. Math. Phys.* **77** (2006) 291 [hep-th/0603130].

[2] T. Buscher, U. Lindström and M. Roček, *New supersymmetric sigma models with Wess-Zumino terms*, *Phys. Lett.* **B 202** (1988) 94.

[3] J. Gates, S. J., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* **B 248** (1984) 157.

[4] M.T. Grisaru, M. Massar, A. Sevrin and J. Troost, *The quantum geometry of $N = (2, 2)$ non-linear sigma-models*, *Phys. Lett.* **B 412** (1997) 53 [hep-th/9706218].

[5] M. Gualtieri, *Generalized complex geometry*, Oxford University DPhil thesis, [math.DG/0401221].

[6] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Q. J. Math.* **54** (2003) no. 3, 281 308, [math.DG/0209099].

[7] N. Hitchin, *Instantons, Poisson structures and generalized Kähler geometry*, *Commun. Math. Phys.* **265** (2006) 131 [math.dg/0503432].

[8] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyperkähler metrics and supersymmetry*, *Commun. Math. Phys.* **108** (1987) 535.

[9] C.M. Hull and B.J. Spence, *The gauged nonlinear sigma model with Wess-Zumino term*, *Phys. Lett.* **B 232** (1989) 204; *The geometry of the gauged sigma model with Wess-Zumino term*, *Nucl. Phys.* **B 353** (1991) 379.

[10] U. Lindström, I.T. Ivanov and M. Roček, *New $N = 4$ superfields and sigma models*, *Phys. Lett.* **B 328** (1994) 49 [hep-th/9401091].

[11] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler geometry and manifest $N = (2, 2)$ supersymmetric nonlinear sigma-models*, *JHEP* **07** (2005) 067 [hep-th/0411186].

[12] U. Lindström, M. Roček, R. von Unge and M. Zabzine, *Generalized Kähler manifolds and off-shell supersymmetry*, *Commun. Math. Phys.* **269** (2007) 833 [hep-th/0512164].

[13] S. Lyakhovich and M. Zabzine, *Poisson geometry of sigma models with extended supersymmetry*, *Phys. Lett.* **B 548** (2002) 243 [hep-th/0210043].

[14] M. Roček, C.-h. Ahn, K. Schoutens and A. Sevrin, *Superspace WZW models and black holes*, hep-th/9110035.

[15] M. Roček, K. Schoutens and A. Sevrin, *Off-shell WZW models in extended superspace*, *Phys. Lett.* **B 265** (1991) 303.

[16] M. Roček and E.P. Verlinde, *Duality, quotients and currents*, *Nucl. Phys.* **B 373** (1992) 630 [hep-th/9110053].

[17] A. Sevrin and J. Troost, *Off-shell formulation of $N = 2$ non-linear sigma-models*, *Nucl. Phys.* **B 492** (1997) 623 [hep-th/9610102].

[18] I. Vaisman, *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics, Vol **118**, Birkhäuser, Basel, 1994.

JHEP04(2007)061